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Results from an Algebraic Classification of Calabi-Yau Manifolds

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Abstract

We present results from an inductive algebraic approach to the systematic construction and classification of the 'lowest-level' CY_3 spaces defined as zeroes of polynomial loci associated with reflexive polyhedra, derived from suitable vectors in complex projective spaces. These CY_3 spaces may be sorted into 'chains' obtained by combining lower-dimensional projective vectors classified previously. We analyze all the 4 242 (259, 6, 1) two- (three-, four-, five-) vector chains, which have, respectively, K3 (elliptic, line-segment, trivial) sections, yielding 174 767 (an additional 6 189, 1 582, 199) distinct projective vectors that define reflexive polyhedra and thereby CY_3 spaces, for a total of 182 737. These CY_3 spaces span 10 827 (a total of 10 882) distinct pairs of Hodge numbers h_{11}, h_{12}. Among these, we list explicitly a total of 212 projective vectors defining three-generation CY_3 spaces with K3 sections, whose characteristics we provide.

1 Introduction

One of the preferred roads towards string phenomenology is via compactification on a Calabi-Yau (CY) manifold [1], either of the perturbative weakly-coupled heterotic string [2], or in some strongly-coupled incarnation such as M or F theory [3]. Most interest has centered on CY manifolds with three complex dimensions (CY_3), but their four-dimensional CY_4 relatives furnish interesting compactifications of F theory, and their two-dimensional K3 relatives are also interesting for illustrative studies. A powerful systematic approach to the construction of CY spaces has been made possible by Batyrev's formulation of them as toric varieties in weighted complex projective spaces, that may be associated with reflexive polyhedra [4]. This work has opened the way for a *Calabi-Yau Genome Project* [5] to classify our possible phenomenological heritage from string theory. In this spirit, an enumeration of all the 473 800 776 reflexive polyhedra that exist in four dimensions, together with a scatter plot of the 30 108 distinct pairs of the Hodge numbers h_{12}, h_{11} of the corresponding CY_3 spaces, has been provided by Kreuzer and Skarke [6].

An informative way to construct such reflexive polyhedra algebraically is to obtain them from vectors in complex projective spaces [7]. These may be built up systematically as discrete linear combinations of vectors in complex projective spaces of lower dimensions, in a way that is universal for all dimensions and naturally gives base-fibre structures. This approach furnishes additional insights into duality, yields relations between spaces in different dimensions obtained by simple geometrical intersection and projection operations, and leads naturally to the appearance of Cartan-Lie algebra structures associated with singularities of the corresponding CY spaces [8, 9]. It also provides illuminating relations between CY spaces in complex projective spaces with the same number of dimensions, that appear in chains of different discrete linear combinations of lower-dimensional vectors. This constructive approach also provides automatically information about fibrations [7], in particular the elliptic and K3 fibrations of CY_3 spaces that are of interest in M and F theory [3].

We have previously illustrated the value added by this systematic algebraic approach to the explicit construction of CY spaces in different dimensions by a discussion of K3 spaces [7]. In particular, we constructed 95 'lowest-level' K3 spaces as zeroes of the polynomial loci associated with projective vectors, and a further 730 as 'higher-level' intersections of such polynomial loci. We also used our algebraic construction to display the corresponding reflexive polyhedra and to illustrate their duality [10], intersection and projection properties, and the association of Cartan-Lie algebra symmetries with singularities of these K3 spaces.

In this paper, we demonstrate the power of our technique in the interesting case of CY_3 spaces, reporting the algebraic constructions of all the spaces obtainable at the lowest level as simple polynomial loci. As a first step, we have surveyed all the CY_3 spaces that belong to the 4 242 chains, identified in our previous paper [7], that can be generated by pairs of projective vectors extended from lower dimensions. Among the 174 767 lowest-level CY_3 spaces found this way, there are 94 distinct projective vectors, with related reflexive polyhedra, that define CY_3 spaces with $h_{11} - h_{12} = N_g$, the number of generations, equal to three. We also find 118 distinct projective vectors corresponding to spaces with $h_{11} - h_{12} = -3$, whose mirrors [11] are three-generation CY_3 spaces. Among these 212 = 94 + 118 three-generation CY_3 spaces there is just one mirror pair. By construction, all of these CY_3 spaces have K3 bases [12, 13, 14, 15]. We tabulate the Hodge numbers of these three-generation CY_3 spaces and the numbers of vertices and edges of the associated reflexive polyhedra and their mirrors, of which there are 179 distinct octuples.

We then go on to explore the CY_3 spaces obtainable from the 259 three-vector chains also identified in our previous paper [7]. These chains yield 6 189 additional lowest-level CY_3 spaces, none of which have $N_g = 3$. All of these spaces, by construction, possess elliptic sections. We complete our construction of the lowest-level CY_3 spaces with those obtainable from the six four-vector chains: 1 582 additional CY_3 spaces, and the one quintuple-vector chain: 199 additional CY_3 spaces. Neither of these sets of chains contribute any new $N_g = 3$ spaces.

In total, this lowest-level algebraic construction produces 10 882 of the 30 108 distinct pairs of CY_3 Hodge numbers h_{12}, h_{11} found by Kreuzer and Skarke [6], and enables us to understand several features of their 'scatter plot' of Hodge numbers: $\chi = 2(h_{11} - h_{12})$ vs $h_{11} + h_{12}$. This plot has diagonal striations that we understand as originating from the series of CY_3 spaces in two-vector chains, and horizontal bands at (roughly) constant $h_{11} + h_{12}$ that result from combinations of these chains. The complete-intersection CY_3 (CICY) spaces correspond to higher-level algebraic constructions in our approach, and results are in preparation [7].

2 Basic Features of the Construction

In the search for CY spaces, one may consider [16] weighted complex projective spaces $CP^n(k_1, k_2, ..., k_{n+1})$, which are characterized by $(n+1)$ quasihomogeneous coordinates $z_1, z_2, ..., z_{n+1}$, with the identification:

$$(z_1, z_2, ..., z_{n+1}) \sim (\lambda^{k_1}.z_1, \lambda^{k_2}.z_2, ..., \lambda^{k_{n+1}}.z_{n+1}) \tag{1}$$

As is well known, the loci of quasihomogeneous polynomial equations in such complex weighted projective spaces yield compact submanifolds, as discussed in more detail in [7] and references therein. Consider a general polynomial $\mathcal{P}$ of degree d:

$$\mathcal{P} \equiv \Sigma_{\vec{\mu}}\, c_{\vec{\mu}} x^{\vec{\mu}} \tag{2}$$

which is a linear combination of monomials $x^{\vec{\mu}} \equiv x_1^{\mu_1} \,.\, x_2^{\mu_2} .....x_{n+1}^{\mu_{n+1}}$ with the condition

$$\vec{\mu} \,.\, \vec{k} = d. \tag{3}$$

The key to our approach is a systematic construction of all possible projective vectors $\vec{k}$, proceeding from lower-dimensional complex spaces to higher-dimensional ones.

We recall that Batyrev [4] found an elegant characterization of CY manifolds in terms of the corresponding Newton polyhedra, defined as the complex hulls of all the vectors $\vec{\mu}$ satisfying (3), which provides a systematic approach to duality [10] and mirror symmetry [11]. Defining the vector $\vec{\mu}' \equiv \vec{\mu} - (1, 1, ..., 1)$, which obeys $\vec{\mu}'.\vec{k} = 0$ and is henceforward denoted without the prime ($\prime$), we define the lattice Λ:

$$\Lambda \equiv \{\vec{\mu} : \vec{\mu} \, . \, \vec{k} = 0\}, \tag{4}$$

with basis vectors $\vec{e}_i$. Consider now the polyhedron Δ, defined as the complex hull of all the vectors $\vec{\mu}$ in Λ which have all components $\mu_i \geq -1$. Batyrev showed that, in order to describe a CY space, such a polyhedron must be reflexive, i.e.,

• the vertices of the polyhedron should correspond to vectors $\vec{\mu}$ with integer components,

• there should be just one interior point, called the center, and

• the distance of any face of the polyhedron from the center should be unity.

He also showed that the mirror Δ^* of any reflexive polyhedron is also reflexive, and hence defines the corresponding mirror CY space.

Kreuzer and Skarke [6] have enumerated the possible reflexive polyhedra in four complex dimensions, and provided a scatter plot of the Hodge numbers of the corresponding CY_3 spaces. Here we reveal some of the algebraic structure of the web of CY_3 spaces, based on relations between the underlying projective vectors $\vec{k}$.

This constructive approach starts [7] from basic projective vectors $\vec{k}_N$ in a lower dimension, and extends them to higher-dimensional projective vectors: $\vec{k}_{N+1} \equiv (0, \vec{k}_N)$ and similarly for the other $(N + 1)$-dimensional vectors obtained by adding the 'zero' coordinate in all the N other possible ways. In the case of K3 spaces, for example, we found [7] a total of 95 such basic projective vectors in four dimensions. These may be extended to 100 different types of projective vector in five dimensions, yielding 10 270 distinct vectors when permutations are taken into account, which we may then use to contruct CY_3 spaces.

To do so, one must combine the basic vectors in all the possible ways that yield distinct reflexive polynomials. In the case of K3 spaces, we found the 95 basic projective vectors by considering 22 two-vector 'chains' (discrete linear combinations of pairs of simple extensions of three-dimensional vectors) and 4 three-vector 'chains', and there was one 'odd vector out' that we found using duality. There was considerable overlap between the different chains: most of the 95 K3 projective vectors appeared in more than one chain, and some appeared in as many as five two-vector chains and three three-vector chains. The topological properties of different spaces in the same chain display systematic relations, as discussed in [7] and again later in this paper.

As we mentioned in [7], the 10 270 distinct basic projective vectors in five dimensions may combined in 4 242 two-vector chains, 259 three-vector chains, 6 four-vector chains and one five-vector chain. The challenge in this approach is to generate all the two- (three-, four-, five-) vector combinations that correspond to distinct reflexive polynomials, and hence CY_3 spaces. In the next Section we outline the results from this construction.

Our method displays immediately interesting fibrations [12, 13, 14]. We recall that the following are equivalent necessary and sufficient conditions for a CY_n space to have as a fibration a CY_{n-k} space [15]: (a) there is a projection operator $\Pi : \Lambda \to \Lambda_{n-k}$, where Λ_{n-k} is an $(n-k)$-dimensional sublattice with $\Pi(\Delta)$ a reflexive polyhedron, or (b) there is a plane of the dual lattice Λ^* through the origin whose intersection with the dual polyhedron Δ^* is an $(n-k)$-dimensional reflexive polyhedron. These conditions are easy to study in our approach, and, as we discuss in the next Section, our two-, three- and four-vector chains naturally correspond to K3, elliptic and reflexive line-segment base-sections.

3 Chains of CY_3 Spaces

3.1 Two-Vector Chains

As already mentioned, there are 4 242 such two-vector chains $m_1\vec{k}^{(1,ext)} + m_2\vec{k}^{(2,ext)}$, where m_1 and m_2 are integers. We call the vector with the lowest values of $m_1 = 1$ and $m_2 = 1$ the eldest vector, and that with the highest value of m_1 and m_2 the youngest vector. Since the maximum values of m_1 and m_2 are restricted by the dimensions of the extended vectors: $m_1 \leq \dim[\vec{k}^{(2,ext)}]$ and $m_2 \leq \dim[\vec{k}^{(1,ext)}]$, it is possible to find in a straightforward but laborious manner all projective vectors in all 4 242 two-vector chains. However, the analogous process becomes prohibitively lengthy in the case of the 259 three-vector chains, so it is desirable to accelerate the process of determining which pairs m_1 and m_2 yield projective vectors that correspond to reflexive polyhedra and hence CY_3 spaces. To this end, we have developed an 'expansion' procedure, which we illustrate in one of the simpler two-vector cases, thereby illuminating some of the geometric features of our construction.

Consider the chain generated by the extended vectors $\vec{k}^{(1,ext)} \equiv (1,0,1,4,6)$, $\vec{k}^{(2,ext)} \equiv (0,1,1,4,6)$, namely $m(1,0,1,4,6)+n(0,1,1,4,6) = (m,n,m+n,4m+4n,6m+6n)[12m+12n]$, where the final number $[N]$ is just the sum of the vector components. All the CY_3 spaces generated by this chain contain [15] a K3 fiber associated with the four-dimensional projective vector $\vec{k}_4 \equiv (1,1,4,6)$, which defines the three-dimensional polyhedron of monomials $\vec{\mu}$ shown in Fig. 1(a). In order to construct the full list of the projective vectors $\vec{k}_5$ in each such two-vector chain generated in our algebraic approach, we 'expand' around a suitable vertex P of the corresponding three-dimensional polyhedron, such as the one shown in Fig. 1(a). This is done by finding all the possible pairs of positive integer points P_1, P_2:

$$P \ = \ \frac{1}{2}(P_1 \ + \ P_2) \tag{5}$$

The set of possible pairs P_1 and P_2 correspond to the set of all the possible integer pairs m_1 and m_2 that determine the two-vector chain. In this way, a reflexive polyhedron of dimension $D = 3$ is expanded into reflexive polyhedra of dimension $D + 1 = 4$. In terms of the expansion point P and the other invariant vertices V_1, V_2, V_3, the allowed new four-dimensional reflexive polyhedra can be described by the projective vectors

$\vec{k}_5$ that satisfy the following chain equations:

$$\begin{aligned}\vec{k}_5 \,.\, P_1 &= \vec{k}_5 \,.\, P_2 \,=\, d, \\ \vec{k}_5 \,.\, V_1 &= \vec{k}_5 \,.\, V_2 \,=\, \vec{k}_5 \,.\, V_3 \,=\, d\end{aligned} \tag{6}$$

This procedure is illustrated in Fig. 1(a) for the K3-fibre chain: $m(1,0,1,4,6) + n(0,1,1,4,6) = (m,n,m+n,4m+4n,6m+6n)[12m+12n]$. We choose as the point of expansion $P \equiv V_4 \equiv (12,12,0,0,0)$, and other three other vertices $V_1 \equiv (0,0,0,0,2), V_2 \equiv (0,0,0,3,0), V_3 \equiv (0,0,12,0,0)$ are left invariant.

The topological invariants of the CY_3 spaces associated with different members of a chain are easily evaluated [7], and we display in Fig. 2(a) the Hodge numbers for the 542 CY_3 spaces in the two-vector chain: $m(0,1,6,14,21) + n(1,0,6,14,21)$. The vectors in this chain have between them 206 distinct values of the Hodge numbers h_{11}, h_{12}, which are visible in Fig. 2(a). There may be equivalences between the CY_3 spaces defined by different projective vectors $\vec{k}_5$ and their corresponding reflexive polyhedra, generated by modular transformations on the coordinates. However, these cannot change the number of points, vertices or edges in either the polyhedron, (N,V,E), or its mirror, (N^*,V^*,E^*). The 542 CY_3 spaces in this two-vector chain have between them 222 distinct values of the sextuple $(N,N^*,V,V^*,h_{11},h_{12})$. There are many other topological invariants that may discriminate further between the 542 CY_3 spaces in this two-vector chain, but a complete exploration of them goes beyond the scope of this paper.

The striations visible in Fig. 2(a) correspond to varying values of m (or n) for some fixed value of n (or m) in the chain $m(0,1,6,14,21)+n(1,0,6,14,21)$. These features reappear in Fig. 2(b), which shows the full set of pairs of Hodge numbers found using all the 4 242 different two-vector chains, which contain a total of 174 767 distinct projective vectors $\vec{k}_5$. By construction, all the CY_3 spaces found in these two-vector chains have K3 base-sections [12, 13, 14, 15], as in the example discussed above. It is interesting to note that this scatter plot contains 10 827 distinct values of the pairs h_{11}, h_{12}. We note that it contains structures reflecting the striations seen in the simple example of Fig. 2(a). This explains the origin of similar structures visible in the scatter plot shown by Kreuzer and Skarke [6], which contains a total of 30 108 different points.

Fig. 3 displays histograms of CY_3 spaces with different values of $h_{11} - h_{12}$ and hence the number of generations $N_g = |h_{11} - h_{12}|$. The CY_3 manifolds of most phenomenological interest are those with $h_{11} - h_{12} = 3$ and their mirror manifolds with $h_{11} - h_{12} = -3$. The larger-scale histogram of Fig. 3(a) is blown up in Fig. 3(b), for a close-up view concentrating on small values of N_g. As already mentioned in the Introduction, we see a total of 94 spaces with $h_{11} - h_{12} = 3$, and a total of 118 with $h_{11} - h_{12} = -3$, whose mirrors have $N_g = 3$. The corresponding projective vectors, Hodge numbers, and relevant properties of the associated reflexive polyhedra are listed in the Tables. Among these, there is just one mirror pair, leaving us with 211 three-generation CY_3 spaces. Between them, these spaces have 99 + 81 - 1 =

179 distinct values of the octuple $(N, N^*, V, V^*, E, E^*, h_{11}, h_{12})$. Thus, we have a total of at least 179 distinct three-generation manifolds at our disposal, all of which possess, by construction, K3 sections. We have not explored systematically the gauge groups that may be associated with their singularities, which would be straightforward in principle. As can be inferred from Figs. 2(a) and 3(b), these three-generation spaces are frequently related to manifolds that are 'nearby' in chains, possibly with 'similar' values of N_g. A systematic study of these features might provide interesting insights into transitions between different CY_3 vacua, but goes beyond the scope of this paper.

Table 1: *Listing of the 94 five-dimensional projective vectors $\vec{k}_5$ defining three-generation CY_3 spaces with K3 sections, including their Hodge numbers and quantities associated with the corresponding reflexive polyhedra.*

$\aleph$	$\vec{k}_5$	N	N^*	h_{12}	h_{11}	V	V^*	$\aleph$	$\vec{k}_5$	N	N^*	h_{12}	h_{11}	V	V^*
1	(2, 3, 8, 11, 17)	62	57	45	48	16	13	2	(2, 5, 9, 10, 11)	37	34	27	30	13	10
3	(2, 5, 14, 21, 33)	65	59	47	50	12	11	4	(3, 4, 5, 8, 13)	38	36	27	30	14	12
5	(3, 4, 10, 15, 17)	37	38	27	30	15	13	6	(3, 5, 15, 16, 24)	39	40	31	34	9	6
7	(3, 5, 16, 24, 45)	60	51	43	46	14	11	8	(4, 5, 6, 15, 21)	37	35	26	29	15	13
9	(4, 5, 7, 8, 9)	24	25	17	20	13	13	10	(4, 5, 7, 9, 10)	24	21	17	20	11	11
11	(4, 6, 7, 9, 17)	26	30	19	22	14	17	12	(3, 7, 8, 10, 25)	37	36	26	29	15	16
13	(3, 5, 6, 8, 17)	37	31	25	28	13	13	14	(2, 7, 9, 10, 13)	34	34	25	28	14	15
15	(2, 8, 9, 11, 21)	40	39	29	32	14	14	16	(3, 4, 6, 13, 13)	38	17	26	29	9	8
17	(5, 8, 9, 11, 12)	18	22	13	16	11	13	18	(2, 7, 8, 11, 17)	38	39	28	31	16	17
19	(4, 5, 7, 10, 13)	25	26	18	21	12	12	20	(2, 6, 9, 17, 17)	43	22	31	34	10	8
21	(2, 8, 9, 21, 23)	44	47	33	36	11	10	22	(2, 7, 12, 15, 21)	40	36	29	32	13	11
23	(2, 8, 9, 27, 37)	64	62	46	49	10	10	24	(2, 8, 21, 41, 51)	62	71	47	50	11	12
25	(2, 11, 12, 39, 53)	66	71	48	51	12	12	26	(5, 6, 9, 14, 17)	22	23	15	18	11	12
27	(3, 7, 8, 9, 10)	25	26	18	21	14	14	28	(4, 5, 6, 21, 27)	46	48	33	36	14	13
29	(4, 6, 15, 35, 45)	44	47	34	37	10	9	30	(4, 5, 8, 10, 17)	29	27	22	25	8	7
31	(2, 5, 11, 20, 27)	59	49	42	45	11	12	32	(2, 5, 17, 26, 33)	64	66	47	50	17	17
33	(3, 4, 17, 27, 30)	54	55	41	44	14	14	34	(3, 4, 21, 32, 39)	64	70	50	53	12	13
35	(5, 6, 8, 33, 47)	50	53	35	38	15	15	36	(3, 5, 14, 20, 21)	36	32	25	28	9	9
37	(4, 5, 11, 29, 38)	47	45	33	36	12	12	38	(5, 6, 7, 29, 40)	47	45	32	35	11	11
39	(3, 6, 10, 19, 35)	46	45	33	36	11	13	40	(2, 5, 11, 20, 33)	68	57	48	51	12	11
41	(2, 5, 13, 22, 37)	72	73	51	54	14	15	42	(2, 5, 17, 26, 45)	80	79	57	60	12	13
43	(2, 5, 23, 32, 57)	94	96	67	70	13	14	44	(4, 5, 5, 18, 22)	43	34	29	32	9	7
45	(3, 9, 10, 11, 24)	33	38	29	32	13	14	46	(3, 9, 14, 15, 16)	29	44	33	36	11	12
47	(5, 7, 9, 15, 36)	32	30	21	24	9	8	48	(4, 5, 7, 18, 20)	34	35	24	27	11	9
49	(5, 6, 9, 10, 21)	27	30	20	23	13	12	50	(1, 10, 12, 13, 15)	58	55	43	46	10	11
51	(1, 11, 13, 14, 16)	58	55	43	46	10	11	52	(5, 6, 9, 12, 13)	21	26	17	20	9	11
53	(4, 5, 7, 8, 11)	25	27	18	21	12	11	54	(1, 8, 13, 14, 17)	63	60	47	50	10	10
55	(1, 14, 23, 24, 31)	63	60	47	50	10	10	56	(1, 11, 18, 19, 24)	63	60	47	50	10	10
57	(4, 5, 10, 11, 19)	29	31	20	23	12	12	58	(4, 4, 9, 11, 17)	32	34	22	25	8	8
59	(4, 5, 8, 11, 17)	28	32	20	23	11	12	60	(1, 5, 11, 16, 18)	84	78	62	65	13	12
61	(1, 8, 18, 27, 29)	84	78	62	65	13	12	62	(1, 9, 20, 30, 33)	84	78	62	65	13	12
63	(1, 8, 10, 21, 23)	81	81	60	63	12	12	64	(5, 6, 7, 8, 13)	22	24	15	18	14	14
65	(1, 8, 11, 20, 23)	77	75	57	60	14	14	66	(3, 12, 14, 21, 25)	29	47	33	36	10	11
67	(2, 9, 11, 24, 35)	55	49	37	40	11	11	68	(3, 5, 8, 22, 33)	58	54	40	43	13	14
69	(1, 8, 10, 21, 31)	97	88	70	73	13	11	70	(4, 5, 7, 20, 31)	46	46	32	35	13	13
71	(3, 4, 15, 38, 57)	98	99	74	77	12	11	72	(3, 8, 30, 79, 117)	98	99	74	77	12	11
73	(2, 9, 12, 23, 37)	51	48	37	40	12	14	74	(3, 4, 13, 24, 41)	67	68	47	50	13	13
75	(4, 5, 7, 23, 34)	51	53	36	39	14	15	76	(5, 6, 14, 45, 65)	56	60	42	45	12	11
77	(4, 7, 13, 41, 58)	53	53	37	40	13	13	78	(5, 8, 12, 15, 35)	30	40	27	30	11	11
79	(2, 7, 15, 18, 21)	40	35	29	32	11	10	80	(2, 9, 19, 24, 27)	40	35	29	32	11	10
81	(3, 7, 15, 18, 20)	32	40	29	32	8	11	82	(3, 8, 18, 21, 25)	32	40	29	32	8	11
83	(1, 15, 22, 28, 33)	63	65	47	50	10	10	84	(1, 18, 26, 33, 39)	63	65	47	50	10	10
85	(1, 21, 30, 38, 45)	63	65	47	50	10	10	86	(1, 11, 16, 20, 23)	63	65	47	50	10	10
87	(1, 13, 19, 24, 28)	63	65	47	50	10	10	88	(1, 16, 23, 29, 34)	63	65	47	50	10	10
89	(7, 10, 11, 14, 35)	26	40	27	30	9	11	90	(3, 8, 21, 43, 54)	52	59	42	45	9	10
91	(4, 5, 26, 65, 95)	91	95	67	70	11	11	92	(2, 9, 26, 65, 93)	100	99	71	74	10	11
93	(2, 9, 30, 73, 105)	109	118	80	83	11	11	94	(10, 12, 13, 15, 25)	16	28	17	20	8	9

Table 2: *Listing of the 118 five-dimensional projective vectors $\vec{k}_5$ defining mirrors of three-generation CY_3 spaces with K3 sections, including their Hodge numbers and quantities associated with the corresponding reflexive polyhedra.*

$\aleph$	$\vec{k}_5$	N	N^*	h_{12}	h_{11}	V	V^*	$\aleph$	$\vec{k}_5$	N	N^*	h_{12}	h_{11}	V	V^*
1	(2, 5, 6, 7, 7)	36	22	26	23	14	10	2	(2, 5, 5, 9, 11)	42	26	30	27	9	7
3	(2, 5, 13, 14, 21)	47	42	35	32	15	15	4	(2, 5, 14, 21, 21)	53	26	38	35	12	9
5	(2, 6, 7, 7, 15)	42	24	30	27	10	8	6	(2, 6, 7, 15, 15)	44	22	32	29	13	10
7	(3, 4, 7, 7, 10)	35	25	23	20	14	10	8	(3, 4, 7, 14, 17)	43	32	30	27	13	11
9	(3, 4, 9, 11, 18)	41	33	33	30	11	11	10	(3, 4, 11, 18, 33)	60	42	43	40	13	10
11	(3, 4, 14, 21, 21)	47	25	35	32	10	8	12	(4, 5, 6, 7, 17)	32	27	23	20	15	15
13	(3, 5, 6, 8, 11)	31	19	21	18	10	10	14	(3, 3, 5, 8, 14)	47	26	31	28	10	9
15	(3, 4, 6, 7, 13)	36	22	25	22	13	12	16	(3, 5, 8, 9, 20)	38	24	26	23	9	9
17	(3, 5, 8, 14, 15)	33	20	23	20	9	9	18	(4, 6, 9, 11, 15)	25	22	22	19	10	9
19	(2, 6, 9, 13, 17)	40	35	30	27	13	15	20	(2, 7, 8, 11, 13)	35	30	26	23	14	14
21	(2, 8, 9, 13, 15)	35	31	26	23	12	13	22	(4, 7, 9, 10, 15)	22	18	16	13	13	11
23	(2, 7, 10, 11, 15)	35	30	26	23	14	13	24	(2, 5, 11, 12, 15)	41	30	30	27	12	12
25	(2, 4, 11, 21, 25)	64	66	49	46	11	11	26	(2, 7, 10, 11, 23)	44	37	32	29	14	14
27	(4, 4, 5, 5, 7)	29	14	20	17	8	7	28	(2, 6, 11, 19, 27)	51	43	38	35	14	15
29	(2, 6, 15, 23, 31)	53	46	40	37	12	14	30	(2, 6, 7, 17, 19)	48	44	36	33	12	12
31	(2, 7, 12, 13, 17)	36	32	27	24	13	13	32	(2, 6, 13, 23, 31)	55	52	41	38	13	12
33	(2, 6, 11, 25, 33)	62	55	46	43	10	11	34	(2, 8, 15, 35, 45)	62	55	46	43	10	11
35	(2, 7, 8, 17, 27)	52	42	38	35	15	16	36	(2, 7, 8, 25, 35)	69	61	50	47	11	12
37	(2, 8, 29, 49, 59)	66	71	51	48	10	11	38	(2, 5, 10, 19, 31)	66	54	47	44	11	10
39	(2, 10, 11, 35, 47)	65	58	47	44	10	11	40	(2, 5, 12, 21, 35)	70	59	50	47	12	12
41	(4, 5, 6, 21, 31)	51	45	36	33	14	12	42	(2, 5, 9, 18, 25)	59	43	42	39	12	11
43	(2, 5, 13, 22, 29)	60	55	44	41	14	16	44	(2, 5, 15, 24, 31)	62	50	45	42	11	11
45	(2, 5, 23, 32, 39)	72	66	53	50	16	17	46	(2, 5, 25, 39, 46)	82	76	60	57	12	10
47	(3, 5, 12, 15, 16)	35	34	31	28	9	8	48	(3, 5, 10, 14, 16)	32	26	23	20	8	7
49	(3, 4, 12, 17, 19)	40	34	29	26	11	12	50	(3, 6, 8, 23, 37)	55	54	40	37	12	12
51	(3, 4, 17, 31, 38)	62	48	45	42	11	12	52	(3, 4, 33, 47, 54)	80	83	63	60	11	12
53	(2, 9, 15, 20, 23)	37	32	27	24	12	11	54	(3, 5, 45, 61, 69)	81	81	63	60	7	6
55	(4, 5, 12, 15, 36)	45	27	36	33	7	6	56	(2, 5, 14, 23, 39)	74	65	53	50	14	13
57	(2, 5, 25, 34, 61)	98	88	70	67	11	11	58	(3, 6, 8, 25, 33)	59	66	54	51	10	11
59	(3, 9, 17, 22, 24)	33	44	38	35	10	12	60	(3, 6, 10, 11, 27)	43	45	38	35	13	13
61	(1, 4, 9, 13, 13)	85	32	62	59	8	7	62	(1, 9, 11, 12, 12)	59	32	43	40	9	7
63	(1, 8, 11, 13, 14)	60	52	45	42	12	12	64	(1, 9, 12, 14, 15)	60	52	45	42	12	12
65	(3, 12, 15, 16, 17)	28	35	32	29	10	11	66	(1, 9, 13, 16, 22)	67	58	50	47	15	15
67	(1, 7, 10, 12, 17)	67	58	50	47	15	15	68	(1, 12, 17, 21, 30)	67	58	50	47	15	15
69	(1, 10, 14, 17, 25)	67	58	50	47	15	15	70	(1, 14, 20, 25, 35)	67	58	50	47	15	15
71	(1, 9, 16, 19, 22)	67	60	50	47	9	9	72	(1, 18, 32, 39, 45)	67	60	50	47	9	9
73	(1, 8, 14, 17, 19)	67	60	50	47	9	9	74	(1, 10, 18, 21, 25)	67	60	50	47	9	9
75	(1, 13, 23, 28, 32)	67	60	50	47	9	9	76	(1, 14, 25, 30, 35)	67	60	50	47	9	9
77	(2, 5, 7, 7, 8)	36	23	26	23	13	9	78	(1, 8, 10, 11, 21)	73	56	53	50	13	13
79	(3, 7, 11, 12, 30)	40	37	32	29	11	12	80	(3, 8, 13, 15, 36)	40	37	32	29	11	12
81	(2, 7, 10, 11, 19)	39	33	29	26	13	14	82	(4, 5, 7, 8, 19)	33	27	23	20	13	11
83	(5, 12, 13, 15, 20)	22	30	26	23	8	11	84	(1, 8, 10, 11, 27)	87	75	63	60	10	10
85	(1, 9, 11, 12, 30)	87	75	63	60	10	10	86	(3, 4, 5, 13, 14)	42	35	31	28	12	13
87	(3, 5, 7, 8, 10)	29	24	21	18	11	12	88	(3, 9, 12, 19, 20)	34	61	50	47	7	9
89	(2, 5, 13, 18, 19)	47	39	35	32	14	13	90	(3, 8, 21, 30, 31)	36	39	32	29	9	10
91	(3, 7, 18, 26, 27)	36	39	32	29	9	10	92	(2, 7, 7, 8, 12)	37	22	30	27	8	7
93	(2, 9, 11, 16, 19)	35	31	26	23	12	12	94	(4, 4, 7, 13, 15)	33	28	23	20	10	9
95	(1, 12, 16, 23, 29)	67	60	50	47	13	13	96	(1, 10, 13, 19, 24)	67	60	50	47	13	13
97	(1, 14, 19, 27, 34)	67	60	50	47	13	13	98	(3, 5, 8, 21, 34)	60	49	41	38	12	12
99	(2, 7, 9, 20, 31)	59	48	41	38	13	13	100	(2, 7, 10, 19, 31)	54	42	39	36	13	13
101	(1, 7, 9, 19, 28)	99	83	71	68	10	9	102	(2, 3, 11, 24, 37)	116	111	83	80	12	12
103	(3, 5, 8, 24, 35)	64	47	43	40	10	9	104	(3, 7, 10, 37, 54)	73	58	49	46	9	10
105	(1, 12, 14, 40, 55)	119	113	94	91	9	10	106	(4, 7, 9, 33, 46)	50	44	35	32	10	11
107	(5, 6, 7, 31, 44)	52	44	36	33	15	13	108	(3, 10, 18, 59, 87)	75	75	60	57	12	11
109	(3, 5, 9, 28, 42)	75	75	60	57	12	11	110	(2, 11, 14, 43, 59)	69	63	50	47	11	12
111	(1, 8, 18, 27, 51)	120	101	86	83	13	11	112	(1, 9, 20, 30, 57)	120	101	86	83	13	11
113	(1, 14, 18, 21, 27)	60	53	45	42	11	10	114	(1, 11, 14, 16, 21)	60	53	45	42	11	10
115	(2, 7, 24, 59, 85)	113	111	81	78	10	10	116	(4, 5, 39, 91, 134)	115	110	81	78	9	10
117	(3, 14, 15, 18, 25)	28	38	32	29	9	11	118	(4, 4, 11, 17, 19)	34	34	24	21	8	8

3.2 Three-Vector Chains

Again as already mentioned, there are 259 three-vector chains $m_1\vec{k}^{(1,ext)}+m_2\vec{k}^{(2,ext)}+m_3\vec{k}^{(3,ext)}$, where m_1, m_2 and m_3 are integers. Also as mentioned earlier, the combinatorial problem of checking each of the possible integer combinations to identify those corresponding to reflexive polyhedra and hence CY_3 spaces necessitated a shortcut solution. This was found in the form of the 'expansion' technique described above, whose extension to the three-vector case we now describe. To find the full list of projective vectors in each of the 259 three-vector chains, one should distinguish two cases. Perhaps (a) one can first find a non-trivial K3 intersection structure from the list of 4 242 two-vector chains by making a simple expansion from a reflexive polyhedron of dimension $D = 2$ to dimension $D + 1 = 3$, and then make a second simple step, as described above, from $D + 1 = 3$ to dimension $D + 2 = 4$. In other cases (b), one makes a double step directly from $D = 2$ to dimension $D + 2 = 4$.

We do not dwell further on the two-step expansion (a), but illustrate the double-step expansion (b) with a worked example. In this case, one first identifies a common planar reflexive polyhedron of monomials $\vec{\mu}$, and then chooses one of its vertices as a 'double-expansion' point P, leaving invariant the other vertices V_i. One then looks for triples P_1, P_2, P_3 of points with two components equal to zero and the property:

$$P = \frac{1}{3}(P_1 + P_2 + P_3) \tag{7}$$

that solve the following chain equations:

$$\begin{aligned} \vec{k}_5 \,.\, P_1 &= \vec{k}_5 \,.\, P_2 = \vec{k}_5 \,.\, P_3 = d, \\ \vec{k}_5 \,.V_1 &= \vec{k}_5 \,.\, V_2 = d, \end{aligned} \tag{8}$$

and then takes the union of these sets.

The 259 three-vector chains may be classified according to which of the 9 different planar reflexive polyhedra, corresponding to CP^2 spaces, appears as a fibre. One particular example is shown in Fig. 1(b): it is the biggest CY_3 chain $m(1,0,0,2,3) + n(0,1,0,2,3) + l(0,0,1,2.3) = (m,n,l,2m+2n+2l,3m+3n+3l)[6m+6n+6l]$, which has an elliptic Weierstrass fibre. There are two fixed points in this chain: $V_1 = (0,0,0,0,2)$ and $V_2 = (0,0,0,3,0)$, and we expand around the point $P = (6,6,6,0,0)$ shown in Fig. 1(b). The eldest vector in this chain is $(1,1,1,6,9)[18]$, and the youngest vector we find is $(91,96,102,578,867)[1734]$. In total, we find 20 796 projective vectors in this particular chain, but only 733 of them are new ones, not found among the 4 242 two-vector chains [1].

The new values of the pairs of Hodge numbers found from another thre-vector chain, namely $m(0,0,1,1,1) + n(0,1,0,1,2) + l(1,0,0,1,2)$, are shown in Fig. 2(c). The total number of distinct new projective vectors, corresponding to new reflexive polyhedra in four dimensions and hence CY_3 spaces, found among all the 259 three-vector chains is 6 189: their Hodge numbers are plotted as circles in Fig. 2(d). We recall that, by construction, these new CY_3 spaces have elliptic sections. Curiously, among all the three-vector chains, we find no new CY_3 spaces with $N_g = 3$.

[1] We recall that a similar overlap was found among K3 spaces appearing in two- and three-vector chains [7].

3.3 Four-Vector Chains

The above methods may be extended to four-vector chains $m_1\vec{k}^{(1,ext)} + m_2\vec{k}^{(2,ext)}$ $+m_3\vec{k}^{(3,ext)} + m_4\vec{k}^{(4,ext)}$, where m_1, m_2, m_3 and m_4 are integers. In this case, one may consider the triple expansion directly from the reflexive line segment in dimension $D = 1$ if one expands about a single point P:

$$P \,=\, \frac{1}{4}(P_1 \,+\, P_2 \,+\, P_3 \,+\, P_4), \tag{9}$$

one must solve the corresponding chain equations:

$$\begin{aligned} &\vec{k}_5 \,.\, P_1 \,=\, \vec{k}_5 \,.\, P_2 \,=\, \vec{k}_5 \,.\, P_3 \,=\, \vec{k}_5 \,.\, P_4 \,=\, d \\ &\vec{k}_5 \,.\, V_2 \,=\, d. \end{aligned} \tag{10}$$

One example is the four-vector chain $(m, n, k, lm+n+k+l)[2m+2n+2k+2l]$, which one may expand around the point $P = (2, 2, 2, 2, 0)$. The eldest vector in this chain is $\vec{k}_5 = (1, 1, 1, 1, 4)[8]$, and the youngest vector is $\vec{k}_5 = (75, 84, 86, 98, 343)[686]$, whilst the total number of projective vectors we find is 14 017. The total number of new projective vectors found in all the 6 four-vector chains is 1 582. The new Hodge numbers of the corresponding CY_3 spaces are shown as crosses in Fig. 2(d).

3.4 The Five-Vector Chain

Finally, we note that the single five-vector chain may be expanded about the point $P = (1, 1, 1, 1, 1)$:

$$P \,=\, \frac{1}{5}(P_1 + P_2 + P_3 + P_4 + P_5). \tag{11}$$

it contains 7 269 projective vectors, of which 199 are new, i.e., not contained among any of the two-, three- or four-vector chains, none of which have $N_g = 3$.

4 Summary and Prospects

We have presented in this paper some first results from our systematic classification of CY_3 spaces, building on our previous study of lower-dimensional projective vectors, reflexive polyhedra and K3 spaces [7]. Our approach gathers CY_3 spaces into chains composed from extensions of lower-dimensional projective vectors, whose fibration structure is explicitly apparent, as illustrated in the two examples exhibited in Fig. 1. We have reported the classification of 182 737 CY_3 spaces spanning 10 882 distinct values of the Hodge numbers h_{11}, h_{12}, as seen in Figs. 2 and 3. Among these, we have reported the projective vectors, Hodge numbers and other properties of 212 CY_3 with three generations and K3 sections, shown in the Table.

It would be interesting to explore these three-generation models in more detail, to see whether any of them are of potential phenomenological interest. The information provided in this paper is sufficient as a starting-point for such a programme of work. The locations of the three-generation models in 'chains' of related CY_3 spaces may eventually provide a tool for understanding transitions between different string vacua, and the dynamical selection of the one we (presumably) occupy. Such a study would require further information beyond that provided in this paper, and we plan to provide the most important data in a forthcoming more detailed paper [17].

As was shown in [7] in the context of K3 spaces, our algebraic approach opens the way to a systematic study of the gauge groups obtainable at the singularities of CY_3 spaces. This is one of the avenues for possible extensions of this work. Other major extensions include the application of our methods to CY_4 spaces, which are of interest in the context of M theory. At the same time, one can construct 'higher-level' CY_3 spaces defined as the intersections of polynomial loci. In the case of K3 spaces, we found more spaces at the higher level than at the lowest level [7], and we expect a similar pattern for CY_3 spaces.

In conclusion, we are optimistic that the results presented here and the prospects they reveal for the future of the 'Calabi-Yau Genome Project' will make possible a much more informed and systematic assault on one of the central problems of string phenomenology, namely the identity and properties of the string vacuum.

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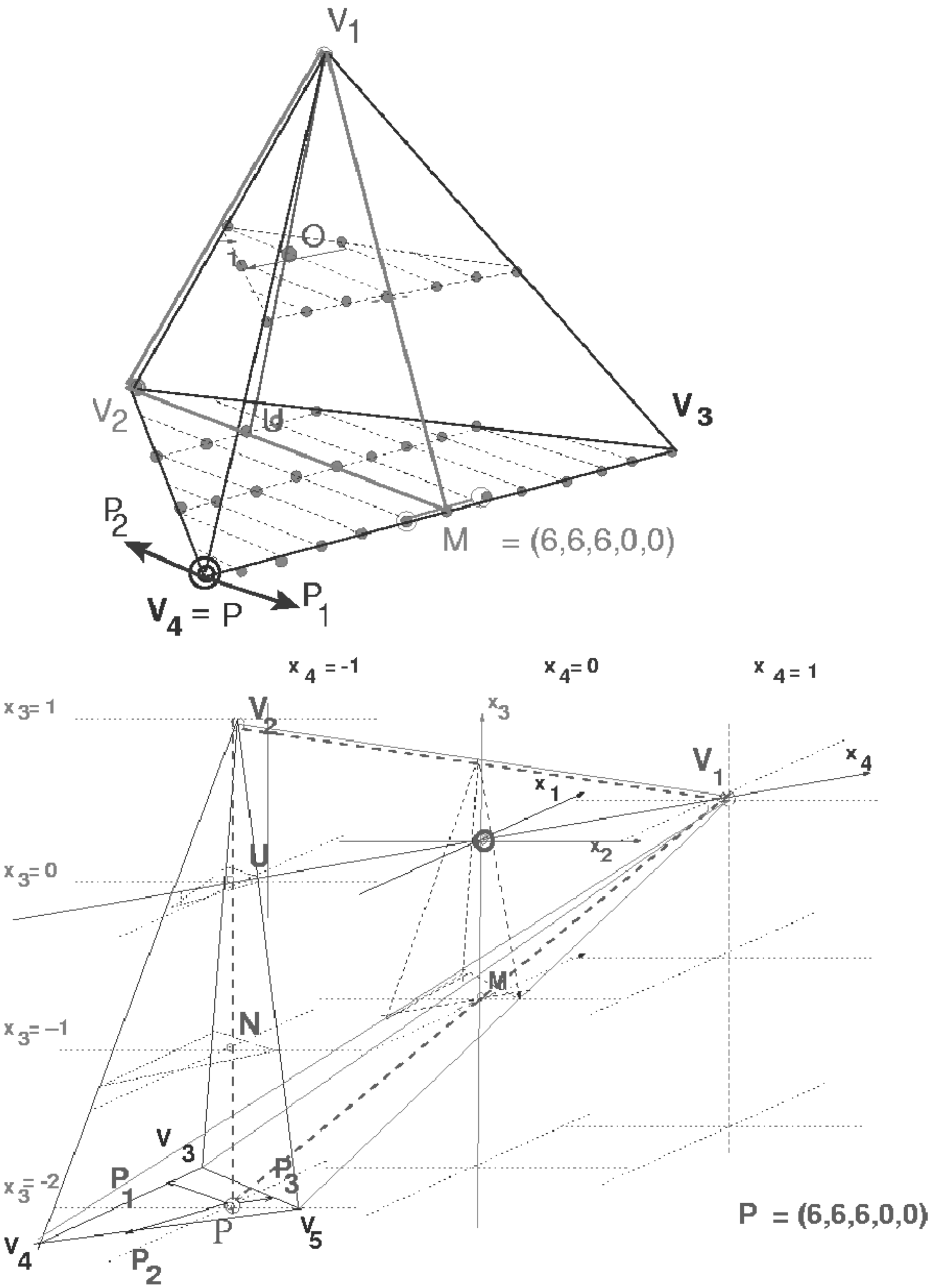

Figure 1: *(a) The 'expansion' method for CY$_3$ spaces in the two-vector chain:* $m(1,0,1,4,6)+n(0,1,1,4,6)$*, which intersect in the K3 space generated by* $k_4 = (1,1,4,6)$ *[7], and (b) the 'double-expansion' method for the three-vector chain* $m(1,0,0,2,3)+n(0,1,0,2,3)+l(0,0,1,2,3)$ *with an elliptic Weierstrass fibre [7], indicated by the dashed triangle. The double-expansion point for this chain is* $P = (6,6,6,0,0)$.

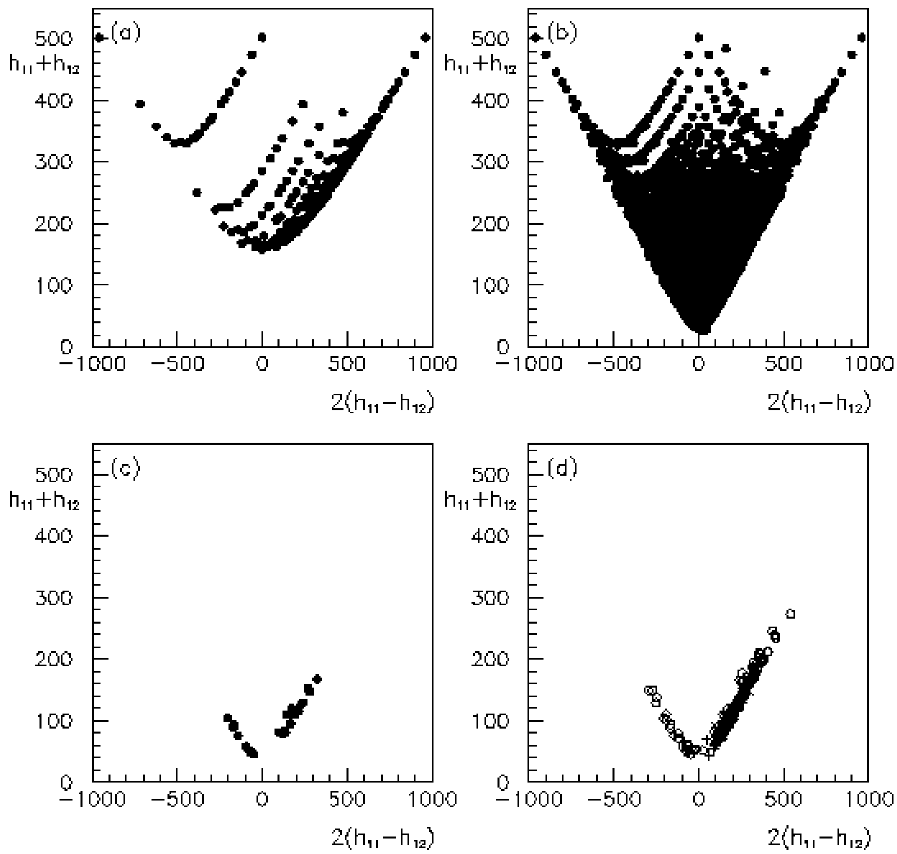

Figure 2: *Scatter plots of the Hodge numbers (a) for all the 542 CY*$_3$ *spaces in one of the 4 242 two-vector chains, namely* $m(0,1,6,14,21)+n(1,0,6,14,21)$*, (b) for all the CY*$_3$ *spaces constructed via two-vector chains, (c) for the additional CY*$_3$ *spaces, not obtainable from any of the two-vector chains, that are found in the three-vector chain* $m(0,0,1,1,1)+n(0,1,0,1,2)+l(1,0,0,1,2)$*, and (d) for all the additional CY*$_3$ *spaces constructed via chains obtained from three (circles) four (crosses) and five vectors (plus signs).*

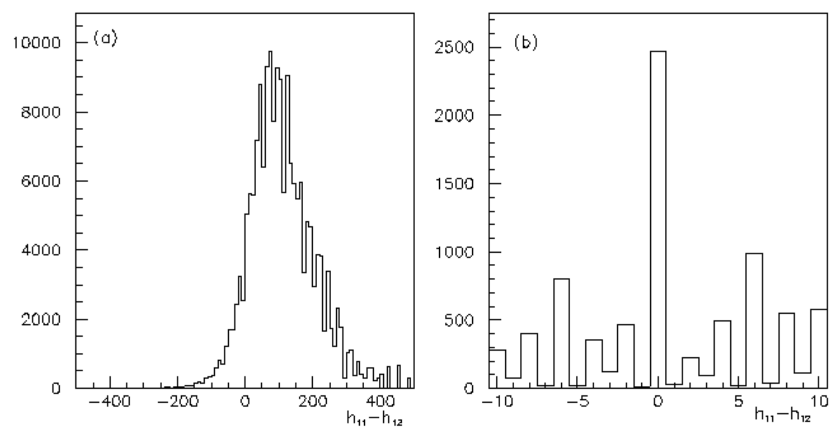

Figure 3: *Histograms of the CY_3 spaces constructed via two-vector chains, as functions of $h_{11} - h_{12}$ and hence the number of generations $N_g = |h_{11} - h_{12}|$, (a) for all the CY_3 spaces found, and (b) for the CY_3 spaces and mirrors with $N_g \leq 10$.*